Fluctuations of Outliers of Finite Rank Perturbations to Random Matrices

David Renfrew

Department of Mathematics
University of California, Davis

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Joint work with A. Soshnikov, A. Pizzo, S. O'Rourke

Wigner Matrices

- Consider a random $N \times N$ Wigner real symmetric matrix

$$X_N = \frac{1}{\sqrt{N}} W_N = \frac{1}{\sqrt{N}} \begin{pmatrix} W_{11} & W_{12} & \dots \\ W_{12} & W_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

W_{ij} are i.i.d. for $1 \leq i < j \leq N$ with

$$\mathbb{E}[W_{12}] = 0, \quad \mathbb{E}[W_{12}^2] = \sigma^2, \quad \mathbb{E}[W_{12}^4] < \infty$$

W_{ii} are i.i.d. for $1 \leq i \leq N$ with

$$\mathbb{E}[W_{11}] = 0, \quad \mathbb{E}[W_{11}^2] < \infty$$

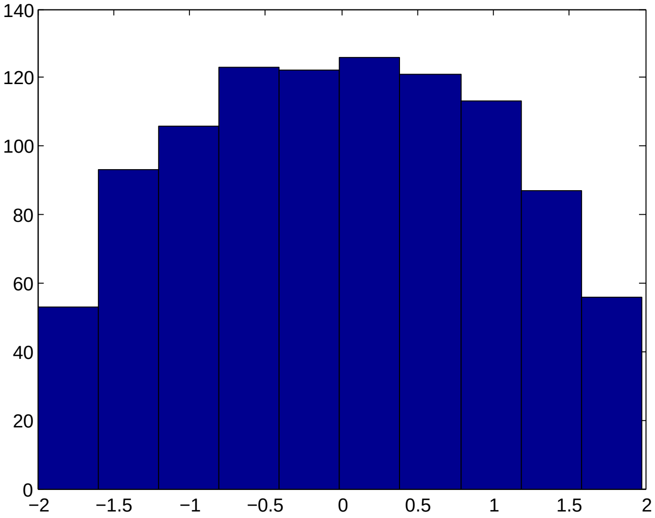
- If $W_{12} \overset{d}{=} \frac{1}{\sqrt{2}} W_{11}$ is Gaussian, then the matrix is said to be from the Gaussian Orthogonal Ensemble (GOE).

Wigner semi-circle law

- The most fundamental result for Wigner matrices is the Wigner semi-circle law.
- A real symmetric matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_N$ induces a measure, called the empirical spectral distribution (ESD), on the real line given by $\frac{1}{N}\sum \delta_{\lambda_i}$.
- The ESD of X_N converges a.s. in distribution to μ_{sc} where

$$\frac{d\mu_{sc}(x)}{dx} = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\mathbf{1}_{[-2\sigma,2\sigma]},$$

  and the largest (smallest) eigenvalue converges to 2σ (-2σ).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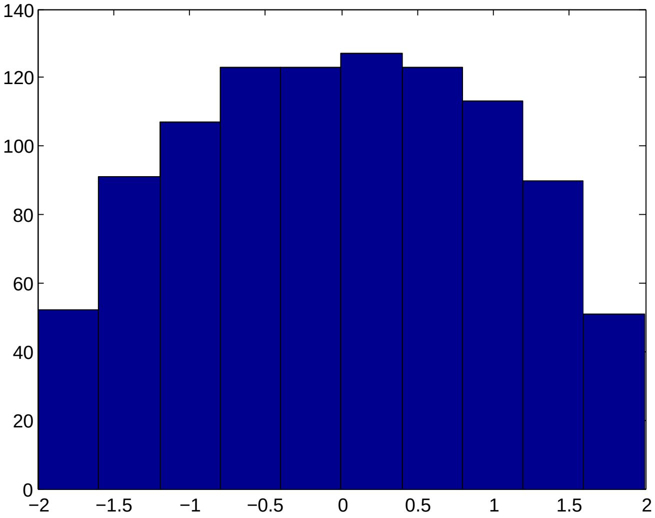
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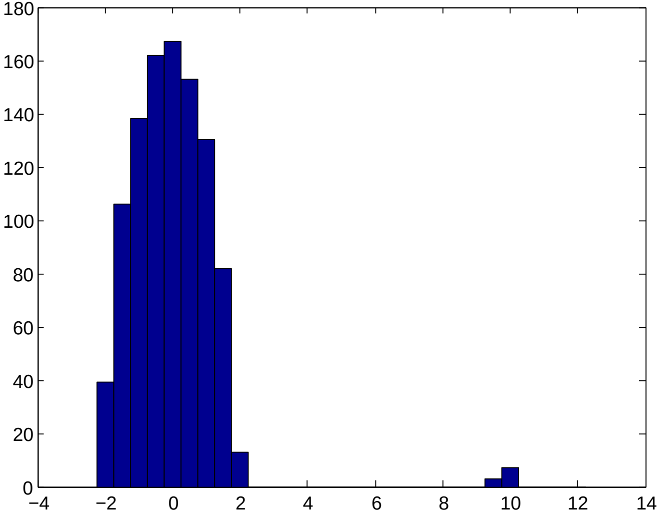
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Wigner semi-circle law

- The Stieltjes Transform, g, of a measure, μ, is given by:

$$g(z) = \int \frac{d\mu(x)}{z - x}.$$

- If μ is an ESD of a matrix, $\mathbf{M}$, then its Stieltjes Transform can be written as

$$\frac{1}{N}\mathrm{Tr}(zI - \mathbf{M})^{-1} =: \mathrm{tr}_N(\mathbf{R}(z)).$$

- The Stieltjes Transform of μ_{sc} satisfies the equation

$$\sigma^2 g_\sigma^2(z) - z g_\sigma(z) + 1 = 0.$$

Deformed Wigner Matrices

- Z. Füredi and J. Komlós ('81) first studied deformed Wigner matrices.
- They assumed the distribution on the entries of the random matrix have a common non-zero mean, c.
- This can be viewed as

$$W_N + C$$

where $(C)_{ij} = c$ is a constant matrix.
- The largest eigenvalue is $Nc + \sigma^2/c$ with Gaussian fluctuations.

Deformed Wigner Matrices

- These results were extended to $W_N/\sqrt{N} + C/N$.
- The largest eigenvalue and the edge of the semicircle are both of constant order.
- First done with Gaussian Matrices by S. Peché ('06).
- Then for Wigner matrices by S. Peché and D. Féral ('07).
- A phase transition is observed depending on the value of c.

Deformed Wigner Matrices

- M. Capitaine, C. Donati-Martin and D. Féral ('09,'12) consider different forms of the perturbation and higher rank perturbations.
- Assume the distribution is symmetric and satisfies a Poincaré Inequality:

$$\mathbb{V}[f(x)] \leq \mathbb{E}[|\nabla f(x)|^2]$$

- Large eigenvalues converge similarly to the rank one case.
- They also show the fluctuations are non-universal for several special perturbations.

Deformed Wigner Matrices

- Concurrent with our research Knowles and Yin also consider finite rank perturbations.
- They assume uniform subexponential decay of the entries but allow the eigenvaules of the perturbation to change with N.
- Give the locations of the outlying eigenvalues for arbitrary finite rank perturbations as well as the distributions when the multiplicities of each eigenvalue of the perturbation is 1.
- Also show that the distribution of the edge eigenvalues stick to the edge eigenvalues of the non-perturbed model and thus have Tracy-Widom fluctuations.

Deformed random matrices

- In this research we consider deformed random matrices given by
$$M_N = X_N + A_N$$
- $A_N = U_N^* \Theta U_N$ has a fixed finite rank and eigenvalues $\{\theta_j\}_{j=1}^r$.
- By the interlacing theorem $N - r$ eigenvalues converge to the semi-circle.
- We are interested in the locations and fluctuations of the remaining r eigenvalues.

Results

Theorem (Pizzo, R., Soshnikov)

Let $J_{+\sigma}(resp.J_{-\sigma})$ be the number of j's such that $\theta_j > \sigma$ (resp., $\theta_j < -\sigma$) and let

$$\rho_j := \theta_j + \frac{\sigma^2}{\theta_j}$$

then:

(a) *For* $1 \leq j \leq J_{+\sigma}, 1 \leq i \leq k_j, \lambda_{k_1+\ldots+k_{j-1}+i} \to \rho_j$

(b) $\lambda_{k_1+\ldots+k_{J_{+\sigma}}+1} \to 2\sigma$

(c) $\lambda_{k_1+\ldots+k_{J-J_{-\sigma}}} \to -2\sigma$

(d) *For* $j \geq J - J_{-\sigma} + 1, 1 \leq i \leq k_j, \lambda_{k_1+\ldots+k_{j-1}+i} \to \rho_j$

the convergence is in probability.

Results

Theorem (Localized case -Pizzo, R., Soshnikov)

$$\left(c_{\theta_j}\sqrt{N}(\lambda_{k_1+\ldots+k_{j-1}+i} - \rho_j),\ i = 1, \ldots, k_j\right)$$

converges in distribution to the distribution of the ordered eigenvalues of V_j.

$$\mathbf{V}_j := \mathbf{U}_j^*(\mathbf{W}_j + \mathbf{H}_j)\mathbf{U}_j,$$

where $\mathbf{W}_j$ is a Wigner random matrix and $\mathbf{H}_j$ is a centered Hermitian Gaussian matrix

$$\mathbb{E}(H_{ss}^2) = \left(\frac{m_4 - 3\sigma^2}{\theta_j^2}\right) + 2\frac{\sigma^4}{\theta_j^2 - \sigma^2},$$

$$\mathbb{E}(|H_{st}|^2) = \frac{\sigma^4}{\theta_j^2 - \sigma^2}.$$

Results

Theorem (Delocalized case -Pizzo, R., Soshnikov)

The difference between

$$\left(c_{\theta_j}\sqrt{N}(\lambda_{k_1+\ldots+k_{j-1}+i}-\rho_j),\ i=1,\ldots,k_j\right)$$

and the vector formed by the (ordered) eigenvalues of a $k_j \times k_j$ GOE (GUE) matrix with the variance of the matrix entries given by

$$\frac{\theta_j^2\sigma^2}{\theta_j^2-\sigma^2}$$

plus a deterministic matrix with entries given by

$$\frac{\theta^2-\sigma^2}{\theta^4}\sum_{i,j} u_i^l \mu_{3,ij} u_j^p$$

converges in probability to zero.

Characterization of outlying eigenvalues

- If z is an eigenvalue of M_N

$$det(zI_N - X_N - A_N) = 0$$

if additionally it is not an eigenvalue of X_N then

$$\begin{aligned}
\det(z - X_N - A_N) &= \det(z - X_N)\det(I + R_N(z)U_N^*\Theta U_N) \\
&= \det(z - X_N)\det(I + \Theta U_N R_N(z) U_N^*) \\
&= \det(z - X_N)\det(\Theta)\det(\Theta^{-1} + U_N R_N(z) U_N^*)
\end{aligned}$$

Resolvent

- We begin with the resolvent identity:

$$zR_N(z) = I_N + X_N R_N(z)$$

$$z\mathbb{E}[R_{ij}(z)] = \delta_{ij} + \sum_l \mathbb{E}[X_{il} R_{lj}(z)]$$

- and use decoupling formula

$$\mathbb{E}(\xi\phi(\xi)) = \sum_{a=0}^{p} \frac{\kappa_{a+1}}{a!} \mathbb{E}(\phi^{(a)}(\xi)) + \epsilon$$

Resolvent

- On the diagonal this becomes

$$\mathbb{E}[R_{ii}(z)] = 1/(z - \sigma^2 g_\sigma(z)) + O(N^{-1}) = g_\sigma(z) + O(N^{-1})$$

- On the off-diagonal this implies

$$\mathbb{E}[R_{ij}(z)] = \frac{\kappa_{3,ij}}{N^{3/2}} g_\sigma^4(z) + o(N^{3/2})$$

- Similarly, we can bound the variance of quadratic forms

$$\mathbb{V}[\mathbf{u}_N^* \mathbf{R}_N(z) \mathbf{v}_N] = O(N^{-1})$$

Resolvent

- Let $\mathbf{u}_N, \mathbf{v}_N$ be a sequence of N dimensional unit vectors.

$$\sqrt{N}\mathbb{E}[\mathbf{u}_N^*\mathbf{R}_N(z)\mathbf{v}_N] - \frac{1}{N}g_\sigma^4(z)\mathbf{u}_N^*\mathbf{M}_3\mathbf{v}_N = \sqrt{N}g_\sigma(z)\mathbf{u}_N^*\mathbf{v}_N + o(1)$$

  where $\mathbf{M}_3 = (1 - \delta_{ij})\kappa_{3,ij}$.

- Furthermore, if $\|\mathbf{u}_N\|_1$ or $\|\mathbf{v}_N\|_1$ is $o(\sqrt{N})$ then the second term on the left side is $o(1)$.

Characterization of outlying eigenvalues

- The eigenvalues are z such that

$$\det(\Theta^{-1} - U_N^* R(z) U_N) = 0.$$

- By the previous estimates and Markov's Inequality

$$\|U_N^* R_N(z) U_N - g_\sigma(z) I_r\| = O(N^{-1/2})$$

with probability going to one.

- Then the eigenvalues converge to

$$g_\sigma^{-1}(1/\theta_k) + O(N^{-1/2}) = \theta_k + \sigma^2/\theta_k + O(N^{-1/2}).$$

- The fluctuations are given by

$$(g_\sigma'(\rho_1) + o(1))(x_i - \rho_1) = -1/\sqrt{N} y_i + o(N^{-1/2})$$

Fluctuations - localized perturbations

Resolvent entries

- Let m be a fixed integer.

$$X_N = \begin{pmatrix} X^{(m)} & B \\ B^* & \tilde{X} \end{pmatrix}$$

$$\tilde{R}(z) = (zI_{N-m} - \tilde{X})^{-1}$$

- By Cramer's rule

$$R^{(m)}(z) = (zI_m + X^{(m)} + B^* \tilde{R}(z) B)^{-1}$$

- Centering, rescaling and then expanding as a geometric series gives:

$$\sqrt{N}(R^{(m)}(z) - g_\sigma(z) I_m) = g_\sigma^2(z)(W^{(m)} + Y_N(z)) + o(1)$$

where:

$$Y_N(z) = \sqrt{N}(B^* \tilde{R}(z) B - \sigma^2 g_\sigma(z) I_m)$$

Central limit for quadratic forms

- Let u_N be an N dimensional vector with entries that are i.i.d. random variables with zero mean and variance one.
- Let A_N be an independent $N \times N$ matrix such that $\|A_N\| < a$ for all N, $\frac{1}{N}\mathrm{Tr}(A_N^2) \xrightarrow{P} a_2$ and $\frac{1}{N}\sum_i A_{ii}^2 \xrightarrow{P} a_1^2$. Then:

$$\frac{1}{\sqrt{N}}\left(u_N^* A_N u_N - \mathrm{Tr}(A_N)\right) \xrightarrow{D} \mathcal{N}(0, \kappa_4 a_1^2 + 2a_2)$$

Central limit for quadratic forms

- This central limit theorem shows that $Y_{ij}(z)$ converges in finite dimensional distributions to a centered Gaussian random variable with covariance:

$$Cov(Y_{ij}(z), Y_{ij}(w)) =$$
$$(1+\delta_{ij})\sigma^2 \frac{g_\sigma(w) - g_\sigma(z)}{z - w} + \delta_{ij}\kappa_4 g_\sigma(z) g_\sigma(w)$$

- The matrix entires $Y_{ij}(z)$ and $Y_{kl}(w)$ are independent up to symmetry.
- Which implies the fluctuations of an eigenvalue with multiplicity k at z are given by the fluctuations of the eigenvalues of:

$$g_\sigma^2(z) U^*(W^{(m)} + G^{(m)}(z))U$$

Theorem (Localized case -Pizzo, R., Soshnikov)

$$\left(c_{\theta_j}\sqrt{N}(\lambda_{k_1+\ldots+k_{j-1}+i} - \rho_j),\ i = 1, \ldots, k_j\right)$$

converges in distribution to the distribution of the ordered eigenvalues of V_j.

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$$\mathbb{E}(H_{ss}^2) = \left(\frac{m_4 - 3\sigma^2}{\theta_j^2}\right) + 2\frac{\sigma^4}{\theta_j^2 - \sigma^2},$$

$$\mathbb{E}(|H_{st}|^2) = \frac{\sigma^4}{\theta_j^2 - \sigma^2}.$$

Fluctuations - delocalized perturbations

Delocalized perturbations

- If $\|u_N^i\|_\infty \to 0$ for all eigenvectors then the fluctuations are universal.

$$(\mathbf{G}_N(z))_{lp} := \sqrt{N}(\mathbf{u}_N^{l*}\mathbf{R}_N(z)\mathbf{u}_N^p - \mathbb{E}[\mathbf{u}_N^{l*}\mathbf{R}_N(z)\mathbf{u}_N^p]).$$

Converges in finite dimensional distributions to $\mathbf{\Gamma}(z)$ with independent, centered, Gaussian entries with covariance given by:

$$\frac{2}{2-\delta_{lp}}\left(-g_\sigma(z)g_\sigma(w) + \frac{g_\sigma(z)g_\sigma(w)}{1-\sigma^2 g_\sigma(z)g_\sigma(w)}\right)$$

for $l \leq p$ and $\Gamma_{lp}(z) = \overline{\Gamma_{pl}(\overline{z})}$ for $l > p$.

Delocalized perturbations

- If we consider $U_N^*(W_N + G_N)U_N$. Where G_N is an $N \times N$ Gaussian matrix with variance as before.

Delocalized perturbations

- Decompose into a Martingale Difference Sequence.

$$\sqrt{N}(\mathbf{u}_N^{l*}\mathbf{R}_N(z)\mathbf{u}_N^p - \mathbb{E}[\mathbf{u}_N^{l*}\mathbf{R}_N(z)\mathbf{u}_N^p]) = \sqrt{N}\sum_k (\mathbb{E}_k - \mathbb{E}_{k-1})\mathbf{u}_N^{l*}\mathbf{R}_N(z)\mathbf{u}_N^p$$

- Apply Martingale central limit theorem.
- Done by Bai and Pan ('12), we extend to non-vanishing third moment and joint distribution of several vectors.

Results

Theorem (Delocalized case -Pizzo, R., Soshnikov)

The difference between

$$\left(c_{\theta_j}\sqrt{N}(\lambda_{k_1+\ldots+k_{j-1}+i}-\rho_j),\ i=1,\ldots,k_j\right)$$

and the vector formed by the (ordered) eigenvalues of a $k_j \times k_j$ GUE (GOE) matrix with the variance of the matrix entries given by

$$\frac{\theta_j^2\sigma^2}{\theta_j^2-\sigma^2}$$

plus a deterministic matrix with entries given by

$$\frac{\theta^2-\sigma^2}{\theta^4}\mathbf{u}_N^l\mathbf{M}_3\mathbf{u}^p$$

converges in probability to zero.

Thank you